

New spectral bounds on the chromatic number combining eigenvalues of the adjacency, Laplacian and signless Laplacian matrices

Clive Elphick* Pawel Wocjan†

November 15, 2012

Abstract

One of the best known results in spectral graph theory is the following lower bound on the chromatic number due to Alan Hoffman, where μ_1 and μ_n are respectively the maximum and minimum eigenvalues of the adjacency matrix: $\chi \geq 1 + \mu_1 / -\mu_n$.

Liliya Kolotilina has demonstrated that $-\mu_n$ in this bound can be replaced with either of two formulae, which involve maximum and minimum eigenvalues of the adjacency, Laplacian and signless Laplacian matrices.

In this paper we use majorization to generalize Kolotilina's bounds, by encompassing all eigenvalues of all three matrices. We compare the performance of these bounds for named and random graphs and derive several known bounds as corollaries of Kolotilina's bounds.

Our proof relies on a technique of converting the adjacency matrix into the zero matrix by conjugating with χ diagonal unitary matrices. We also prove that the minimum number of diagonal unitary matrices that can be used to convert the adjacency matrix into the zero matrix is the normalized orthogonal rank, which can be strictly less than the chromatic number.

1 Introduction

Let G be a graph with n vertices, m edges, chromatic number χ and adjacency matrix A . Let D be the diagonal matrix of vertex degrees. Let $L = D - A$ denote the Laplacian of G and $Q = D + A$ denote the signless Laplacian of G .

The eigenvalues of A are denoted by $\mu_1 \geq \dots \geq \mu_n$; of L by $\theta_1 \geq \dots \geq \theta_n = 0$; of Q by $\delta_1 \geq \dots \geq \delta_n \geq 0$. It is known that for all graphs $\delta_i \geq 2\mu_i$ holds for $i = 1, \dots, n$.

*clive.elphick@gmail.com

†Mathematics Department & Center for Theoretical Physics, Massachusetts Institute of Technology, Boston, USA; on sabbatical leave from Department of Electrical Engineering and Computer Science, University of Central Florida, Orlando, USA; wocjan@eecs.ucf.edu

In 1970 Hoffman [9] proved that:

$$\chi \geq 1 + \frac{\mu_1}{|\mu_n|}. \quad (1)$$

In 2007 Nikiforov [12] modified Hoffman's bound with the following hybrid bound involving the maximal eigenvalues of the adjacency matrix and the Laplacian matrix:

$$\chi \geq 1 + \frac{\mu_1}{\theta_1 - \mu_1}. \quad (2)$$

In 2011 Kolotilina [10] improved Nikiforov's bound with the following hybrid bound involving the maximal eigenvalues of the adjacency matrix, the Laplacian and the signless Laplacian:

$$\chi \geq 1 + \frac{\mu_1}{\mu_1 - \delta_1 + \theta_1}. \quad (3)$$

This bound is never worse than Nikiforov's bound.

Kolotilina also proved the following bound:

$$\chi \geq 1 + \frac{\mu_1}{\mu_1 - \delta_n + \theta_n}. \quad (4)$$

Observe that we include $\theta_n = 0$ on purpose to emphasize that our new bounds naturally extend this bound.

We generalize Kolotilina's bounds by including all eigenvalues of the three matrices. Our proof relies on techniques from majorization theory and provides an alternative and more intuitive proof of the special cases established by Kolotilina.

Theorem 1.1. *The chromatic number is bounded from below by*

$$\chi \geq 1 + \frac{\sum_{i=1}^m \mu_i}{\sum_{i=1}^m (\mu_i - \delta_i + \theta_i)} \quad (5)$$

$$\chi \geq 1 + \frac{\sum_{i=1}^m \mu_i}{\sum_{i=1}^m (\mu_i - \delta_{n+1-i} + \theta_{n+1-i})} \quad (6)$$

for all $m = 1, \dots, n$.

Remark 1.2. *In 2011 Lima, Oliveira, Abreu and Nikiforov [11] proved that*

$$\chi \geq 1 + \frac{2m}{2m - n\delta_n}. \quad (7)$$

We provide a new and simpler proof of this result in Section 3 below. Observe that since $\mu_1 \geq 2m/n$, bound (4) follows immediately from this result.

2 Mathematical tools

2.1 Majorization of spectra of self-adjoint matrices

We recall some basic definitions and results in majorization. We refer the reader to [1, Chapters II and III]. Let $x = (x_1, \dots, x_n)$ be an element of \mathbb{R}^n . Let x^\downarrow be the vector obtained by rearranging the coordinates of x in the non-increasing order. Thus, if $x^\downarrow = (x_1^\downarrow, \dots, x_n^\downarrow)$, then $x_1^\downarrow \geq \dots \geq x_n^\downarrow$.

Let $x, y \in \mathbb{R}^n$. We say that x is majorized by y , in symbols $x \prec y$, if

$$\sum_{i=1}^m x_i^\downarrow \leq \sum_{i=1}^m y_i^\downarrow \quad (8)$$

for $m = 1, \dots, n-1$ and

$$\sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow.$$

Let X be an arbitrary self-adjoint matrix of size n with complex entries and $\lambda^\downarrow(X)$ denote the vector in \mathbb{R}^n whose coordinates are the eigenvalues of X sorted in non-increasing order. We refer to $\lambda^\downarrow(X)$ as the spectrum of X .

The statement of the following lemma is presented in [1, Problem I.6.15]:

Lemma 2.1. *Let X be a self-adjoint matrix of size n with complex entries. For all $m = 1, \dots, n$, we have*

$$\sum_{i=1}^m \lambda_i^\downarrow(X) = \max \sum_{i=1}^m v_i^\dagger X v_i, \quad (9)$$

$$\sum_{i=1}^m \lambda_{n+1-i}^\downarrow(X) = \min \sum_{i=1}^m v_i^\dagger A v_i, \quad (10)$$

where the maximum and minimum are taken over all choices of m orthonormal (row) vectors v_1, \dots, v_m in \mathbb{C}^n . The first statement is referred to as the Ky Fan's Maximum Principle. It is also formulated in [1, Exercise II.1.13].

Lemma 2.2. *Let X and Y be two arbitrary self-adjoint matrices of size n . For $m = 1, \dots, n$, the sum of the m largest eigenvalues of $X + Y$ is bounded from below and above by:*

$$\sum_{i=1}^m \lambda_i^\downarrow(X) + \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(Y) \leq \sum_{i=1}^m \lambda_i^\downarrow(X + Y) \leq \sum_{i=1}^m \lambda_i^\downarrow(X) + \sum_{i=1}^m \lambda_i^\downarrow(Y) \quad (11)$$

Let U and V be two arbitrary self-adjoint matrices of size n . For $m = 1, \dots, n$, the sum of the m smallest eigenvalues of $U + V$ is bounded from above and below by:

$$\sum_{i=1}^m \lambda_i^\downarrow(U) + \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(V) \geq \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(U + V) \geq \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(U) + \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(V) \quad (12)$$

Proof. The right hand inequality in (11) follows from [1, Corollary III.4.2], which shows that

$$\lambda^\downarrow(X + Y) \prec \lambda^\downarrow(X) + \lambda^\downarrow(Y). \quad (13)$$

The right hand inequality in (11) is also equivalent to the claim made in [1, Exercise II.1.14].

To establish the left hand inequality in (11), we choose the orthonormal vectors v_1, \dots, v_n such that $Xv_i = \lambda_i^\downarrow(X)v_i$. Using the results (9) and (10) of the previous lemma, we obtain

$$\sum_{i=1}^m \lambda_i^\downarrow(X + Y) \geq \sum_{i=1}^m v_i^\dagger (X + Y) v_i = \sum_{i=1}^m \lambda_i^\downarrow(X) + \sum_{i=1}^m v_i^\dagger Y v_i \geq \sum_{i=1}^m \lambda_i^\downarrow(X) + \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(Y).$$

Note that the left hand inequality in (11) also follows from [1, Equation (III.13)].

The upper and lower inequalities in (12) follow from (11) by setting $X = -V$ and $Y = -U$ and applying the identities $\lambda_i^\downarrow(-U) = -\lambda_{n+1-i}^\downarrow(U)$ and $\lambda_i^\downarrow(-V) = -\lambda_{n+1-i}^\downarrow(V)$. \square

2.2 Conversion of the adjacency matrix A into the zero matrix

Recall that G is colorable with c colors if there exists a map $\Phi : V = \{1, \dots, n\} \rightarrow C = \{1, \dots, c\}$ such that $a_{k\ell} = 1$ implies $\Phi_k \neq \Phi_\ell$ for all $k, \ell \in V$. In words, the graph can be colored with c colors if it is possible to assign at most c different colors to its vertices such that any two adjacent vertices receive different colors. The chromatic number χ is the minimum number of colors required to color the graph.

The following theorem was proved by Wocjan and Elphick [14].

Theorem 2.3 (Conversion of adjacency matrix). *Assume that there exists a coloring of G with c colors. Then, there exist c diagonal unitary matrices U_1, \dots, U_c whose entries are c th roots of unity such that*

$$\sum_{s=1}^c U_s^\dagger A U_s = 0. \quad (14)$$

Without loss of generality we may assume that U_c is the identity matrix. We obtain the following corollary.

Corollary 2.4. *Assume that there exists a coloring of G with c colors. Let A be the adjacency matrix of G and D an arbitrary diagonal matrix. Then, we have*

$$\lambda^\downarrow\left(D + \frac{1}{c-1}A\right) \prec \lambda^\downarrow(D - A). \quad (15)$$

Proof. Theorem 2.3 implies that

$$\sum_{s=1}^c U_s^\dagger (D - A) U_s = cD$$

since $U_s D U_s^\dagger = D U_s U_s^\dagger = D$ holds because the diagonal matrices U_s and D commute and $U_s U_s^\dagger = I$ for all s . We may assume without loss of generality that the last matrix U_c is equal to the identity matrix, which leads to the matrix equation

$$\sum_{s=1}^{c-1} U_s^\dagger (D - A) U_s = (c - 1)D + A.$$

We obtain the desired statement by dividing both sides by $(c - 1)$ and then applying the majorization inequality (13). We also use that conjugation by a unitary matrix does not change the spectrum of a self-adjoint matrix. \square

Note that the above result contains

$$\lambda_1^\downarrow \left(D + \frac{1}{c-1} A \right) \leq \lambda_1^\downarrow (D - A)$$

as the special case, which was proved by Nikiforov in [12, Theorem 1] using entirely different techniques.

We now strengthen the statement of Theorem 2.3. We show that the parameter c with $c \geq \chi$ can be replaced by d with $d \geq \xi'$, where ξ' is the normalized orthogonal rank of G .

Orthogonal representations of graphs have been investigated in [3, 7] in the study of the quantum chromatic number. A d -dimensional orthogonal representation of G is a map $\Psi : V \rightarrow \mathbb{C}^d$, mapping vertices to d -dimensional column vectors such that $a_{k\ell} = 1$ implies $\langle \Psi_k, \Psi_\ell \rangle = 0$ for $k, \ell \in V$, where $\langle \Psi_k, \Psi_\ell \rangle = \Psi_k^\dagger \Psi_\ell$ denotes the inner product between these two column vectors.

The orthogonal rank of G , denoted by ξ , is the minimum d such that there exists an orthogonal representation of G in \mathbb{C}^d . Furthermore, let ξ' be the smallest d such that G has an orthogonal representation in the vector space \mathbb{C}^d with the added restriction that the entries of each vector must have modulus one [3]. We refer to these representations as normalized orthogonal representations and to ξ' as the normalized orthogonal rank.

It can be shown that $\xi' \leq \chi$ and that there are graphs whose normalized orthogonal rank is strictly smaller than their chromatic number [14, Subsection 2.4].

In the following it is convenient to assume that all the vectors of an orthogonal representation have length d .

Lemma 2.5. *Let U_1, \dots, U_d be a collection of diagonal unitary matrices of size n and use the notation*

$$U_s = \text{diag}(u_{1,s}, \dots, u_{n,s}) \quad \text{for } s = 1, \dots, d \tag{16}$$

to denote the diagonal entries of the matrices U_s .

Then, we have

$$\sum_{s=1}^d U_s^\dagger U_s = I \quad \text{and} \quad \sum_{s=1}^d U_s A U_s^\dagger = 0 \tag{17}$$

if and only if the vectors

$$\Psi_k = (u_{k,1}, \dots, u_{k,d})^T \quad \text{for } k = 1, \dots, n$$

form a normalized orthogonal representation of the graph with adjacency matrix A .

Proof. The entry on the left hand side of (17) in the k th row and ℓ th column is equal to

$$\sum_{s=1}^d \bar{u}_{k,s} u_{\ell,s} a_{k,\ell} = \langle \Psi_k, \Psi_\ell \rangle a_{k,\ell},$$

where equality follows from the definition of the vectors Ψ_k . \square

This discussion demonstrates that the lower bounds on the chromatic number are in fact lower bounds on the normalized orthogonal rank.

Remark 2.6. *Note that if we use vectors of an orthogonal representation that is not normalized, then we also obtain $d \geq \xi$ diagonal matrices D_1, \dots, D_d satisfying the following two properties:*

$$\sum_{s=1}^d D_s^\dagger D_s = I \quad \text{and} \quad \sum_{s=1}^d D_s A D_s^\dagger = 0.$$

In contrast, we may not assume without loss of generality that one of these matrices is equal to the identity matrix and, thus, we cannot convert A to $-A$. The latter is necessary in order to prove lower bounds on the orthogonal rank with majorization techniques.

3 Proof of the new hybrid bound

Proof. First, we apply Corollary 2.4 to the matrix $L = D - A$ and obtain

$$\lambda^\downarrow \left(D + \frac{1}{\chi - 1} A \right) \prec \lambda^\downarrow(D - A) = \lambda^\downarrow(L),$$

which implies the inequalities

$$\sum_{i=1}^m \lambda_i^\downarrow(L) \geq \sum_{i=1}^m \lambda_i^\downarrow \left(D + \frac{1}{\chi - 1} A \right) \quad (18)$$

for all $m = 1, \dots, n$.

Second, we rewrite

$$D + \frac{1}{\chi - 1} A = (D + A) - pA = Q - pA \quad \text{where} \quad p = \frac{\chi - 2}{\chi - 1}. \quad (19)$$

Third, we apply the left hand inequality in (11) in Lemma 2.2 with $X = Q$ and $Y = -pA$, which results in the inequalities

$$\sum_{i=1}^m \lambda_i^\downarrow(Q - pA) \geq \sum_{i=1}^m \lambda_i^\downarrow(Q) + p \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(-A) \quad (20)$$

Finally, combining (18), (19) and (20) and applying the identities $\lambda_{n+1-i}^\downarrow(-A) = -\lambda_i^\downarrow(A)$ to the rightmost sum in (20) we obtain

$$\sum_{i=1}^m \lambda_i^\downarrow(L) \geq \sum_{i=1}^m \lambda_i^\downarrow(Q) - p \sum_{i=1}^m \lambda_i^\downarrow(A).$$

This inequality is equivalent to the bound in (5) in Theorem 1.1.

The bound in (6) in Theorem 1.1 is proved similarly. For the sake of completeness we spell out all the steps.

First, we apply Corollary 2.4 to the matrix $Q = D + A$ and obtain

$$\lambda^\downarrow\left(D - \frac{1}{\chi - 1}A\right) \prec \lambda^\downarrow(D + A) = \lambda^\downarrow(Q),$$

which implies the inequalities

$$\sum_{i=1}^m \lambda_{n+1-i}^\downarrow(Q) \leq \sum_{i=1}^m \lambda_{n+1-i}^\downarrow\left(D - \frac{1}{\chi - 1}A\right) \quad (21)$$

for all $m = 1, \dots, n$.

Second, we rewrite

$$D - \frac{1}{\chi - 1}A = (D - A) + pA = L + pA \quad \text{where} \quad p = \frac{\chi - 2}{\chi - 1}. \quad (22)$$

Third, we apply the left hand inequality in (12) in Lemma 2.2 with $U = pA$ and $V = L$, which results in the inequalities

$$\sum_{i=1}^m \lambda_{n+1-i}^\downarrow(L + pA) \leq \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(L) + p \sum_{i=1}^m \lambda_i^\downarrow(A) \quad (23)$$

Finally, combining (21), (22) and (23) we obtain

$$\sum_{i=1}^m \lambda_{n+1-i}^\downarrow(Q) \leq \sum_{i=1}^m \lambda_{n+1-i}^\downarrow(L) + p \sum_{i=1}^m \lambda_i^\downarrow(A).$$

This inequality is equivalent to the bound in (6) in Theorem 1.1. \square

Remark 3.1. Let \tilde{D} be an arbitrary diagonal matrix. Then, we obtain valid bounds on the chromatic number by replacing the Laplacian L and signless Laplacian Q by the modified Laplacian $\tilde{L} = \tilde{D} - A$ and signless Laplacian $\tilde{Q} = \tilde{D} + A$, respectively. A further generalization is possible by considering weighted adjacency matrices $W * A$ (where $*$ denotes the Schur product) for arbitrary self-adjoint matrices W and arbitrary diagonal matrices \tilde{D} .

Proof. We now give a simple proof of the Lima, Oliveira, Abreu and Nikiforov bound in (7). Using the conversion technique, the identity $D - Q = -A$, and the invariance of the diagonal entries under conjugation by the diagonal unitary matrices U_s we obtain

$$A = \sum_{s=1}^{c-1} U_s(-A)U_s^\dagger = \sum_{s=1}^{c-1} U_s(D - Q)U_s^\dagger = (c-1)D - \sum_{s=1}^{c-1} U_sQU_s^\dagger.$$

Define the column vector $v = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^T$. We multiply the left and right most sides of the above matrix equation by v^\dagger from the left and by v from the right and obtain

$$\frac{2m}{n} = v^\dagger Av = (c-1)\frac{2m}{n} - \sum_{s=1}^{c-1} v^\dagger U_sQU_s^\dagger v \leq (c-1)\frac{2m}{n} - (c-1)\delta_n.$$

We have used here that $v^\dagger Av = v^\dagger Dv = 2m/n$, which is equal to the sum of all entries of respectively A and D divided by n due to the special form of v , and $w^\dagger U_sQU_s^\dagger w \geq \lambda_{\min}(U_sQU_s^\dagger) = \delta_n$ holds for arbitrary unit vectors w due to (10) in Lemma 2.1 and due to the invariance of the spectrum of Q under conjugation by the unitary U_s . \square

4 Performance of the bounds

We can derive several known bounds from Theorem 1.1 by using various properties of Laplacian and signless Laplacian eigenvalues, which are for example proved in section 3.9 of Brouwer and Haemers [2]. In particular we use that $\delta_i \geq 2\mu_i$ and $\theta_1 \leq n$ for all graphs.

For example:

$$\chi \geq 1 + \frac{\mu_1}{\mu_1 - \delta_1 + \theta_1} \geq 1 + \frac{\delta_1}{2\theta_1 - \delta_1} \geq 1 + \frac{\delta_1}{2n - \delta_1} \geq 1 + \frac{\mu_1}{n - \mu_1}.$$

The second bound in this chain is inequality 3.17 in Kolotilina [10]; the third bound is due Hansen and Lucas [6]; and the fourth bound is due to Cvetkovic [4]. Similarly:

$$\chi \geq 1 + \frac{\mu_1}{\mu_1 - \delta_1 + \theta_1} \geq 1 + \frac{\mu_1}{\theta_1 - \mu_1} \geq 1 + \frac{\mu_1}{n - \mu_1}.$$

The second bound in this chain is due to Nikiforov [12].

Wilf [13] proved that Cvetkovic's bound is in fact a lower bound for the clique number and recently He, Jin and Zhang [8] and de Abreu and Nikiforov (in a paper entitled "Clique number and the signless Laplacian" which has been submitted for publication) proved that Hansen and Lucas's bound is also a lower bound for the clique number.

Theorem 1.1 and the Hoffman bound are, of course, not lower bounds for the clique number as evidenced for example by the triangle-free Grötzsch graph.

4.1 Extremal graphs

The bounds in Theorem 1.1 are exact for complete graphs and for complete regular multipartite graphs. They are also exact for all bipartite graphs, because the spectra of L and Q are identical for all bipartite graphs and because Desai and Rao [5] have proved that $\delta_n = 0$ if and only if a graph has a bipartite component. The bounds in Theorem 1.1 (with $m = 1$) are equal to Hoffman's bound for regular graphs, and therefore these bounds are exact for all regular graphs for which Hoffman's bound is exact. A coloring for which Hoffman's bound is exact is called a Hoffman coloring (see section 3.6 of [2]).

4.2 Named graphs

The Wolfram function `GraphData[n]` generates parameters for named graphs on n vertices. It is useful to compare four lower bounds for the chromatic number:

- the Hoffman bound in (1)
- the generalization of the Hoffman bound in Wocjan and Elphick [14] with $m \geq 2$
- Theorem 1.1 with $m = 1$
- Theorem 1.1 with $m \geq 2$.

We have examined Wolfram's named graphs on 16, 25 and 28 vertices. Based on this limited sample, the results are as follows.

For all regular graphs, Theorem 1.1 equals the Hoffman bound and for most regular graphs this bound is not improved by setting $m \geq 2$ in either Theorem 1.1 or the generalization of Hoffman. There are, however, exceptions. For example, for `Circulant(16, (1,7,8))`, bound (5) with $m = 1$ and Hoffman equal 2.7; but bound (5) with $m = 3$ and the generalization of Hoffman in Wocjan and Elphick [14] with $m = 3$ equals 2.9. The chromatic number of this graph is 4.

For irregular graphs, the position is more diverse. There are graphs for which all four bounds are best, with the Hoffman bound being best most frequently. For example, the `NoPerfectMatchingGraph` on 16 vertices has Hoffman equal to 2.5; bound (5) with $m = 1$ equal to 2.7; Hoffman with $m = 3$ equals 2.8; and bound (5) with $m = 3$ equal to 2.9. The chromatic number of this graph is 4. `Barbell(8)` has Hoffman equal to 4.8 but bound (6) with $m = 1$ equal to 7.3. `Sun(8)` has Hoffman equal to 4.1 but bound (5) with $m = 1$ equal to 5.5.

4.3 Random Graphs

The Wolfram function `RandomGraph[n,p]` generates a random graph $G_{n,p}$ on n vertices with each edge being present with independent probability p . Eigenvalues are found using the function `Spectrum`, provided the Wolfram package "Combinatorica" has been loaded.

Theorem 1.1 with $m \geq 2$ almost never exceeds Theorem 1.1 with $m = 1$ for random graphs, because for almost all random graphs $\mu_1 \gg \mu_2$.

Tabulated below is the performance of bounds (3) and (4) against the Hoffman bound for each combination of $n = 20$ and 50 and $p = 0.5, 0.7$ and 0.9 , in each case averaged over 15 graphs. We have included a comparison with the Bollobas result that the chromatic number of almost every random graph $G_{n,p}$ is:

$$q = (1/2 + o(1))n / \log_b(n),$$

where $b = 1/(1 - p)$.

n	p	Hoffman	Bound (3)	Bound (4)	Bollobas
20	0.5	3.5	3.1	2.7	2.3
20	0.7	4.4	4.1	3.3	4.0
20	0.9	6.5	7.8	4.7	7.7
50	0.5	4.6	4.0	3.4	4.4
50	0.7	6.2	5.3	4.5	7.7
50	0.9	10.1	9.9	6.6	14.7

It can be seen that for $n = 20$ all bounds exceed the Bollobas formula for varying levels of p , because for low levels of n the $o(1)$ term is material. The main conclusion is that bound (3) outperforms the Hoffman bound for smaller, denser random graphs. On average, bound (4) is consistently less good than bound (3), but for individual graphs bound (4) sometimes exceeds bound (3). Bounds (3) and (4) are sometimes better and sometimes worse than the conjectured lower bound in Wocjan and Elphick [14], which involves the sum of the squares of the eigenvalues of the adjacency matrix.

5 Conclusions

In this paper and in Wocjan and Elphick [14], we have demonstrated how lower bounds for the chromatic number can often be generalized to encompass all eigenvalues, which for some graphs improves their performance.

We have also discussed lower bounds for the clique number which are not lower bounds for the chromatic number. These bounds cannot be generalized in an equivalent way, because for example, the bound due to He et al [8] would monotonically increase to $n/(n - m)$.

The proof of Theorem 1.1 is straightforward because of the power of combining Theorem 2.3 with majorization, and because the proof uses graph matrices rather than the eigenvectors of these matrices.

Our proof that the minimum number of diagonal unitary matrices that can be used to convert the adjacency matrix into the zero matrix is the normalized orthogonal rank, may enable results to be proved about the normalized orthogonal rank that could not be proved about the chromatic number.

Acknowledgements

We would like to thank Vladimir Nikiforov, in private correspondence, for drawing our attention to the paper by Kolotilina, which was published in Russian in 2010 and translated into English in 2011.

P.W. gratefully acknowledges the support from the National Science Foundation CAREER Award CCF-0746600. This work was supported in part by the National Science Foundation Science and Technology Center for Science of Information, under grant CCF-0939370.

References

- [1] R. Bhatia, *Matrix analysis*, Graduate text in mathematics, vol. 169, Springer
- [2] A. Brouwer and W. Haemers, *Spectra of graphs*, Universitext, Springer, 2012; www.win.tue.nl/~aeb/2WF02/spectra.pdf
- [3] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini, A. Winter, *On the quantum chromatic number of graph*, The Electronic Journal of Combinatorics, 14, #R81, 2007.
- [4] D. Cvetkovic, *Chromatic number and the spectrum of a graph*, Publ. Inst. Math. (Beograd), 14(28) (1972), 25–38.
- [5] M. Desai and V. Rao, *A characterization of the smallest eigenvalue of a graph*, J. Graph Theory 18 (1994), 181–194.
- [6] P. Hansen and C. Lucas, *An inequality for the signless Laplacian index of a graph using the chromatic number*, Graph Theory Notes N.Y., 57 (2009), 39-42.
- [7] G. Haynes, C. Park, A. Schaeffer, J. Webster, L. H. Mitchell, *Orthogonal vector coloring*, The Electronic Journal of Combinatorics, 17, #R55, 2010; <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v17i1r55/>
- [8] B. He, Y-L. Jin and X-D. Zhang, *Sharp bounds for the signless Laplacian spectral radius in terms of clique number*, Linear Algebra Appl. (2011), doi:10.1016/j.laa.2011.10.008
- [9] A. J. Hoffman, *On eigenvalues and colourings of graphs*, in: Graph Theory and its Applications, Academic Press, New York (1970), pp. 79–91.
- [10] L. Yu. Kolotilina, *Inequalities for the extreme eigenvalues of block-partitioned Hermitian matrices with applications to spectral graph theory*, Journal of Mathematical Sciences, vol. 176, no. 1, July 2011.
- [11] L. S. de Lima, C. S. Oliveira, N. M. M. de Abreu, V. Nikiforov, *The smallest eigenvalue of the signless Laplacian*, Linear Algebra and its Applications, vol. 435, issue 10, (2011), 2570 - 2584.

- [12] V. Nikiforov, *Chromatic number and spectral radius*, Linear Algebra Appl. 426, 810-814, 2007.
- [13] H. Wilf, *Spectral bounds for the clique and independence numbers of graphs*, J. Combin. Theory Ser. B 40(1986), 113–117.
- [14] P. Wocjan and C. Elphick, *New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix*, arXiv:1209.3190 (2012).